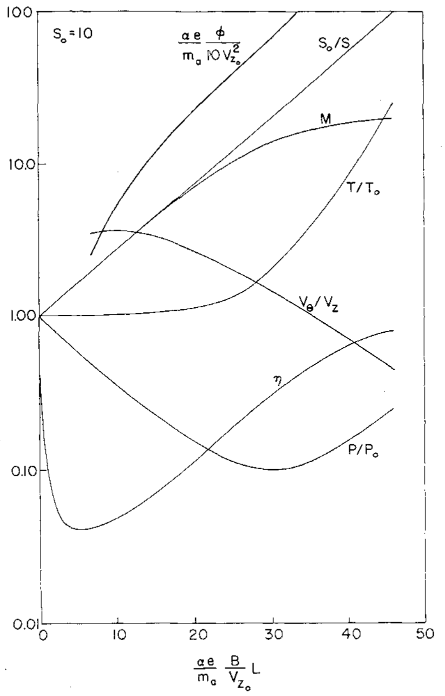

Fig. 3 Accelerator parameters for $S_0 = 10$.

sulting equations for Mach number, voltage, accelerator length, and azimuthal velocity in terms of (S_0/S) are

$$M^2 = \frac{1 + [(\gamma - 1)/(\gamma + 1)](1/S_0^2)}{(S/S_0)^2 + [(\gamma - 1)/(\gamma + 1)](1/S_0^2)} \tag{11}$$

$$\frac{\alpha e}{m_a}\phi = \frac{1}{2}V_{z_0}^2\left(1 + \frac{\gamma - 1}{\gamma}\frac{1}{S_0^2}\right)\left(1 + \frac{\gamma - 1}{\gamma + 1}\frac{1}{S_0^2}\right)^{-1} \times \left\{\left[\left(\frac{S_0}{S}\right)^2 - 1\right] + (S_0^2 + 1) \times \left(1 + \frac{\gamma - 1}{\gamma + 1}\frac{1}{S_0^2}\right)^{-1}\left(\ln\frac{S_0}{S}\right)^2\right\} \tag{12}$$

$$\frac{\alpha e}{m_a}\frac{B}{V_{z_0}}z = S_0\left(1 + \frac{\gamma - 1}{\gamma}\frac{1}{S_0^2}\right)\left(1 + \frac{\gamma - 1}{\gamma + 1}\frac{1}{S_0^2}\right)^{-1}\ln\frac{S_0}{S} \tag{13}$$

and

$$\frac{V_\theta}{V_z} = S_0\left(1 + \frac{1}{S_0^2}\right)\left(1 + \frac{\gamma - 1}{\gamma + 1}\frac{1}{S_0^2}\right)^{-1}\frac{S}{S_0}\ln\frac{S_0}{S} \tag{14}$$

At $(S/S_0) = 0$, corresponding to an infinite velocity ratio, the Mach number is finite and depends only on S_0. For $b_0 > 0$, however, both velocity ratio and Mach number are finite. Constant Mach number contours are shown in Fig. 1 for $b_0 = 0$. The linear dependence of V_θ on z can be obtained by eliminating (S_0/S) from Eqs. (13) and (14).

Finally an integrated conversion efficiency, representing the ratio of axial body force work to electrical energy input[1] was found to be

$$\eta = \frac{\{1 + [(\gamma - 1)/\gamma](1/S_0^2)\}[(S_0/S)^2 - 1] - (1/\gamma)\ln(S_0/S)^2}{[1 + (1/S_0^2)]\{[(S_0/S)^2 - 1] + (1 + S_0^2) \times \{1 + [(\gamma - 1)/(\gamma + 1)](1/S_0^2)\}^{-1}[\ln(S_0/S)]^2\}} \tag{15}$$

At S_0, $\eta = (\gamma - 1/\gamma)$. It can be shown from the electron energy equation that the remaining $(1/\gamma)$ of the electrical energy input appears as initial "electron heating."[6] Inspection of Eq. (15) indicates that given (S_0/S), there is an optimum value of S_0 that maximizes η. As (S_0/S) increases, the optimum S_0 increases. This is associated with the ratio of azimuthal to axial accelerating force which can be expressed as[1]

$$\frac{J_z B}{J_\theta B} = \frac{1 + \beta S}{\beta - S} \tag{16}$$

Thus $J_z B > J_\theta B$ for $S > (\beta - 1)/(\beta + 1)$ and the major portion of the body force work appears as rotational energy. This is apparent from Figs. 2 and 3, which show the variation of accelerator parameters with length for $\gamma = \frac{5}{3}$ and $S_0 = 1$ and 10, respectively. For $S_0 = 10$, η is low until S decreases to appreciably less than 1, whereas for $S_0 = 1$, there is almost a steady rise until a limiting efficiency determined by Joule heating is reached.

The constant degree of ionization analysis described in this note corresponds to negligibly small ionization and recombination rates. Actual accelerator characteristics can be bracketed by considering infinite rates in which case, the degree of ionization is in equilibrium with the local pressure, ion-atom temperature, and electron temperature. This requires consideration of the electron energy equation[6] and a suitable Saha-type equation.[7] Since α would increase due to the static temperature rise shown in Figs. 2 and 3, βS would increase during acceleration according to Eq. (7).

As $\alpha \rightarrow 1$ the form of Ohm's Law used to derive Eqs. (1, 2, and 7) must be modified to include electron pressure gradients. These gradients act to decouple the thermodynamic from the body force effects, particularly at large ratios of electron to ion-atom temperatures.

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Matrizant of Keplerian Motion

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In a recent paper[1] on the matrizant of Keplerian motion (or the "error matrix" or "guidance matrix," etc.), I was rash enough to comment that there was no general formula applicable to every type of orbit, except over time spans short enough for series expansions in the time to be used. I would have done better to have written that I was not, at the time, aware of any such formula, for I rapidly became indebted to W. H. Goodyear[2] for sending me one. Goodyear's formulas

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are related to those of Bower (see Ref. 1), but he uses Herrick's variables[3] that are valid for any eccentricity.

Several authors have recently published formulas for this matrix, and I have hesitated before adding further to the proliferation. The purpose of this note is to give two forms that are simpler than those I gave earlier and can be used in any type of orbit. They can also be used to advantage in the integration of perturbed orbits.[4]

It is required to find relations between small departures in position and velocity from Keplerian motion at two different times, t and t_0. These departures can be related to a single set of small changes in any six independent geometrical elements of the orbit. Let $\delta\mathbf{W}$ be a column matrix containing the departures in position and velocity at time t, and $\delta\mathbf{W}_0$ the same for time t_0. Let $\mathbf{G}$ be a set of six independent geometrical elements. Then

$$\delta\mathbf{W} = \mathbf{M}(t)\delta\mathbf{G} = \mathbf{M}(t)\mathbf{M}^{-1}(t_0)\delta\mathbf{W}_0 \tag{1}$$

One of the simplest forms of $\mathbf{M}$ occurs when $\delta\mathbf{W}$ is resolved into rectangular Cartesian coordinates, the X axis pointing toward perihelion and the Z axis along the angular momentum vector of the orbit (this was referred to as the "orbital reference system" in Ref. 1), and also when

$$\delta\mathbf{G}^T = [\delta l_0 + \delta\mathrm{r}, \delta\mathrm{p}, \delta\mathrm{q}, e\delta\mathrm{r}, \delta a/a, \delta e]$$

Here, l_0 is the mean anomaly at an arbitrary epoch; δp, δq, δr are infinitesimal rotations about the reference axes; a is the semimajor axis; and e the eccentricity of the Keplerian orbit. This set, and the corresponding matrix, were given by Eckert and Brouwer[5, 6]; only the deviations in position were considered, but those in velocity can be found very easily from differentiation with respect to the time. I note, in passing, that the quantities $(HX + KX')$ and $(HY + KY')$, introduced by Eckert and Brouwer, can, in this instance, be written as $a(YX'/h - 1)$ and $-aXX'/h$, respectively, where h is the angular momentum per unit mass in the orbit, and a prime stands for differentiation with respect to the time.

Now let the members of $\delta\mathbf{G}$ be modified so that

$$\delta\mathbf{G}_1{}^T = [(\delta l_0 + \eta\delta\mathrm{r})/n, a\delta e/h, \delta\mathrm{p}, \delta a/2a, ae\delta\mathrm{r}/h, \delta\mathrm{q}]$$

where n is the mean motion, and $\eta = (1 - e^2)^{1/2}$. Then, eschewing the algebra, which is straightforward, we find

$$\mathbf{M} = \begin{bmatrix} X' & YX' - h & 0 & 2X - 3tX' & YY' & 0 \\ Y' & -XX' & 0 & 2Y - 3tY' & -YX' - 2h & 0 \\ 0 & 0 & Y & 0 & 0 & -X \\ X'' & Y'X' + YX'' & 0 & -X' - 3tX'' & Y'^2 + YY'' & 0 \\ Y'' & -X'^2 - XX'' & 0 & -Y' - 3tY'' & -X'Y' - YX'' & 0 \\ 0 & 0 & Y' & 0 & 0 & -X' \end{bmatrix}$$

$$\mathbf{M}^{-1} = \mathbf{A\Phi M}^T\mathbf{\Phi}^T \tag{2}$$

where

$$\left.\begin{aligned} \mathbf{A} &= [a/\mu \quad a/\mu h \quad 1/h \quad a/\mu \quad a/\mu h \quad 1/h]\mathbf{I} \\ \mathbf{\Phi} &= \begin{bmatrix} \mathbf{O} & -\mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{bmatrix} \end{aligned}\right\} \tag{2a}$$

$\mathbf{I}$ being the identity matrix of relevant dimension. The accelerations are given by $X'' = -\mu X/R^3$, etc.; μ is the appropriate constant of attraction for the problem, and t is the time measured from an arbitrary epoch. The components of position and velocity can be found using separate formulas for special orbits or universal formulas, but only this one formula is needed for the matrizant.

For some purposes it is convenient to use displacements resolved along the following directions: δR along the radius vector, δT in the forward transverse direction in the plane of the orbit, and δZ defined as previously. In formulas (3) and (4) the quantities $\delta R'$ and $\delta T'$ represent the velocity displacement resolved along the R and T directions, and not the errors in the radial and transverse components of velocity. We have

$$\begin{bmatrix} \delta R \\ \delta T \\ \delta Z \\ \delta R' \\ \delta T' \\ \delta Z' \end{bmatrix} = \mathbf{NN}^{-1} \begin{bmatrix} \delta R_0 \\ \delta T_0 \\ \delta Z_0 \\ \delta R_0' \\ \delta T_0' \\ \delta Z_0' \end{bmatrix} \tag{3}$$

where

$$\mathbf{N} = \begin{bmatrix} R' & -hX/R & 0 & 2R - 3tR' & -hY/R & 0 \\ h/R & -RX' + hY/R & 0 & -3ht/R & -RY' - hX/R & 0 \\ 0 & 0 & Y & 0 & 0 & -X \\ -\mu/R^2 & hX'/R & 0 & -R' + 3\mu t/R^2 & hY'/R & 0 \\ 0 & -R'X' - RX'' & 0 & -h/R & -R'Y' - RY'' & 0 \\ 0 & 0 & Y' & 0 & 0 & -X' \end{bmatrix}$$

$$\mathbf{N}^{-1} = \mathbf{A\Phi N}^T\mathbf{\Phi}^T \tag{4}$$

as before. R and R' refer to the radial distance from the attracting focus and its time derivative. The X and Y coordinates and their derivatives are, in the derivation, components referred to the orbital reference system. But it is not hard to show that in the second set of formulas they can be components referred to any pair of rectangular axes fixed in the plane of the orbit (because, in the product $\mathbf{NN}_0{}^{-1}$, X, Y, and their derivatives occur in forms such as $XX_0 + YY_0$ or $XY_0' - YX_0'$, etc., which can be put into vectorial forms such as $\mathbf{R}\cdot\mathbf{R}$ or $\mathbf{R} \times \mathbf{R}_0'$, etc., and so are independent of the reference system). Therefore, these formulas can lead to no difficulties when the eccentricity is very small.

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